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FINITE ELEMENT ANALYSIS OF CUT-GROWTH IN SHEETS OF HIGHLY ELASTIC MATERIALS

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Abstract—The Rivlin–Thomas criterion for the tearing of rubber vulcanizates is discussed in terms of Rice's path-independent J integral. It is shown that the R–T criterion can be stated in an alternative form where the growth of a cut is governed by this integral. The J criterion is then used to predict the critical load that causes a cut to grow in the uniaxial stretching of nicked rubber vulcanizate strips. Calculation of the J integral is accomplished with the use of a finite element procedure for highly elastic materials. It is shown that the predicted critical loads agree fairly closely with existing experimental data.

1. INTRODUCTION

IT HAS been shown by Rivlin and Thomas [1] that the tearing of rubber vulcanizate is governed by a Griffith-type energy criterion; namely, that a cut in a rubber sheet will spread if the rate of decrease of elastic energy stored in the sheet with respect to the size of the cut reaches a critical energy characteristic of the material. The criterion has been employed in subsequent investigations [2–5] to determine the effect of the shape of the cut, of the temperature, and of the dynamic cut-growth rate on the energy characteristic and, more recently, in studies of fatigue properties of rubbers [6–8]. In structural analysis applications, however, the Rivlin–Thomas criterion has seldom been utilized, for example, in predicting the rupture load of rubber-like materials with existing flaws. One major difficulty has been the computation of the stress and strain distributions in a solid body which undergoes a finite deformation. Moreover, if the existing flaws are of the crack kind, the strain concentration near the tip of the crack causes further difficulty. The objective of this paper is to explore the usefulness of the finite element method in predicting the critical load that causes the growth of a cut in highly elastic materials. Much of the work is motivated by recent advances made in finite element procedures for problems of finite elastic deformation (e.g. Oden [9]), and by the recent application of the Rice integral [10] in elasto–plastic crack analyses (Broberg [11] and Andersson [12]).

The content of this paper may be summarized as follows: In Section 2 we give a discussion of the relationship between the Rivlin-Thomas criterion and the Rice integral. It is shown that the critical energy characteristic of Rivlin and Thomas can be used to define a critical value of the Rice integral. This leads to an alternative criterion for the tearing of rubbers in terms of the Rice integral. In Section 3, an analysis is made of the uniaxial stretching of several nicked rubber vulcanizate strips, using a finite element procedure for highly elastic materials. The finite element procedure is discussed in detail in Appendix A. Calculations are made to determine the relationship between the load and the Rice integral for several lengths of cut. From these calculations and from the known energy characteristics of these vulcanizates, the critical load that causes a cut to grow is predicted as a function of the cut length. In Section 4, we compare the calculated critical loads with experimental data of Rivlin and Thomas [1], and find good agreement between calculation and experiment.

2. THE RIVLIN-THOMAS CRITERION AND THE RICE INTEGRAL

The criterion of Rivlin and Thomas [1] for the growth of a cut in rubber sheets is expressed by the following equation:

$$-\left(\frac{\partial W}{\partial c}\right)_l = T_c h, \tag{2.1}$$

where W denotes the elastic energy stored in the sheet, c is the cut length measured off in the undeformed state, h is the undeformed thickness of the sheet, and T_c is the critical energy characteristic of the material. The suffix l denotes that the differentiation is carried out under conditions of constant displacement of the part of the boundary which is not traction free. Thus, the quantity $(\partial W/\partial c)_l$ is equivalent to the rate of change of the potential energy P with respect to the cut length, or $\mathrm{d}P/\mathrm{d}c$. To determine T_c, Rivlin and Thomas have used a "simple extension" tear test piece and a "pure shear" test piece (shown in Fig. 1). For both types of test pieces, analytical formulae have been derived for the quantity $-(\partial W/\partial c)_l$ in terms of overall force and deformation variables (equations (6.8) and (6.10) in [1]).

We now proceed to discuss the relationship between the Rice integral [10] and the Rivlin–Thomas criterion. For a homogeneous body containing a traction-free cut subject

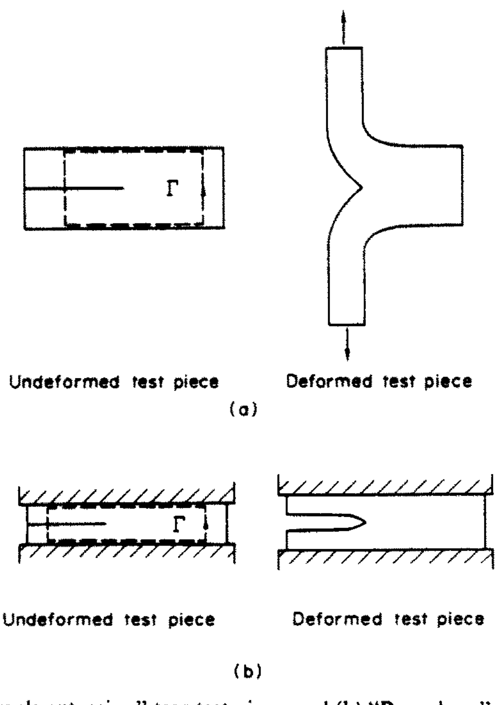


FIG. 1. (a) "Simple extension" tear test piece, and (b) "Pure shear" tear test piece.

to a two-dimensional infinitesimal deformation field, the Rice integral is defined by

$$J = \int_{\Gamma} \left[W \,\mathrm{d}x^2 - \mathbf{T} \cdot \frac{\partial \mathbf{u}}{\partial x^1} \,\mathrm{d}s \right], \tag{2.2}$$

where W denotes the strain energy density, $\mathbf{T}$ and $\mathbf{u}$ denote the traction and displacement vectors, respectively. The integral assumes the same value for all paths Γ surrounding the tip of the cut (see Fig. 2). For finite deformation, the Rice integral J may still be written in

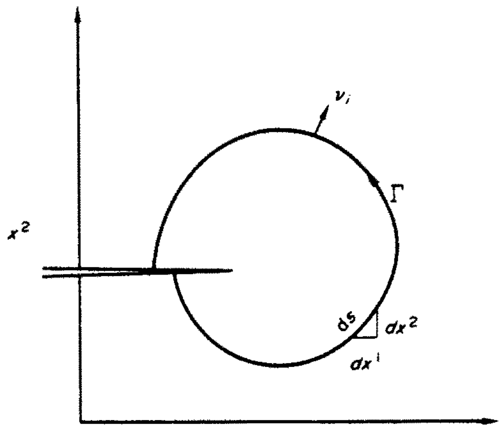


FIG. 2. Notation for defining the Rice integral.

the form of (2.2) provided that W represents the strain energy density per unit initial (undeformed) volume, $\mathbf{T}$ is the nominal traction vector, x^1 and x^2 are the initial cartesian coordinates, and $\mathrm{d}s$ is the differential arc length along Γ, defined in the undeformed geometry. To prove this, we need only to show that the integral so defined is path independent. Without loss of generality, we assume the body to be a sheet of initial thickness h, and express the Rice integral J by

$$J = h \int_{\Gamma} \left[W \,\mathrm{d}x^2 - t^{ij} v_i \frac{\partial u_j}{\partial x^1} \,\mathrm{d}s \right], \qquad (i, j = 1, 2), \tag{2.3}$$

where t^{ij} are contravariant components of stress vectors, resolved with respect to the initial base vectors of (x^1, x^2) and measured per unit area of the undeformed geometry, v_i denote the outward normal of Γ, and u_i are displacement components with respect to x^i. The strain energy density W (per unit initial volume) is defined by

$$W = \int_0^{\varepsilon_{ij}} s^{ij} \,\mathrm{d}\varepsilon_{ij}, \tag{2.4}$$

where s^{ij} are contravariant components of the Kirchhoff stress (measured per unit area of the undeformed geometry) on convected coordinates initially coincide with (x^1, x^2), and ε_{ij} are finite strain tensors defined by

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + u^k_{,i} u_{k,j}). \tag{2.5}$$

Here, we have used the convention that a comma preceding an index denotes partia differentiation. The stresses t^{ij} are related to s^{ij} by

$$t^{ij} = s^{ik}(\delta^j_k + u^j_{,k}) \tag{2.6}$$

(see p. 6 of [13]). Substituting (2.5) and (2.6) in (2.4) gives an alternative form for W:

$$W = \int_0^{u_{j,i}} t^{ij} d(u_{j,i}). \tag{2.7}$$

With the use of equation (2.7) and the equilibrium equations $t^{ij}_{,i} = 0$ for finite deformation it can easily be verified that the integral (2.3) is path independent.

It has been shown by Rice [14] that for elastic materials, linear or nonlinear, the Rice integral J is identical to $-\mathrm{d}P/\mathrm{d}c$, the rate of potential energy decrease with respect to the cut size. Rice distinguished two types of cut-tip configurations, smooth-ended and sharp ended, and proved the identity

$$J = -\frac{\mathrm{d}P}{\mathrm{d}c}, \tag{2.8}$$

for both configurations. Although Rice's proofs were given in the context of infinitesima deformation, it can be shown that they are valid also for finite deformation as long as the Rice integral J (2.2) is defined in its finite deformation form (e.g. (2.3)). Consequently, the identity (2.8) holds for both infinitesimal and finite deformations and for both smooth ended and sharp-ended tip configurations. Since the quantity $(\partial W/\partial c)_l$ of Rivlin and Thomas is actually $\mathrm{d}P/\mathrm{d}c$, it follows from equation (2.8) that $J = -(\partial W/\partial c)_l$.

We now discuss specifically the Rice integral (2.3) for both the "simple extension" tear test piece and the "pure shear" test piece of [1]. For the simple extension test piece we evaluate equation (2.3) following the contour Γ as sketched in Fig. 1(a). The only portions of Γ along which J does not vanish are seen to be those perpendicular to the cut The state of deformation there corresponds to a uniform extension λ subject to a force F if the initial cross-sectional area is hl_0, l_0 being the total width of the specimen, the first term of the Rice integral (2.3) is $-Whl_0$, while the second term gives $t^{12}hl_0(\partial u_2/\partial x^1$ which equals to $2F\lambda$. Comparing this result with equation (6.8) of [1] confirms the identity $J = -(\partial W/\partial c)_l$. For the pure shear test piece, we evaluate equation (2.3) following the contour as shown in Fig. 1(b). Comparing the resulting expression with (6.10) of [1] lead again to the desired identity.

From the above discussions, it is evident that the critical energy characteristic, T_c determined by Rivlin and Thomas using the two test pieces may be construed such that i in fact defines a critical value of the Rice integral, J_c (apart from a constant factor h), which can now be used in an alternative criterion for the tearing of rubber sheets expressed in terms of J. The J criterion may be stated as follows: A cut in a rubber sheet will spread i the Rice integral J reaches a critical value J_c $(=T_ch)$. We conclude this section by noting that the use of the J integral for crack stability criteria in elasto–plastic fracture mechanic has been discussed by Broberg [11].

3. UNIAXIAL STRETCHING OF NICKED RUBBER VULCANIZATE STRIPS

In this section we apply the J criterion discussed in the previous section to predic the critical load that causes a cut to grow in the uniaxial stretching of nicked rubbe vulcanizate strips. The geometry of the undeformed strips is of thickness h and of width b with a cut of length c (see Fig. 3). Three vulcanizates are investigated. The recipes and material properties of these vulcanizates were given in the Appendix and Table 4 of Ref. [1], as labeled by vulcanizates A, D and E. The rubber vulcanizates are assumed to be incompressible, and their strain energy function satisfies the Mooney form:

$$W = C_1[(I_1-3)+\alpha(I_2-3)],$$

where C_1 and α are material constants, and I_1, I_2 are strain invariants (see (A.15) of Appendix A).

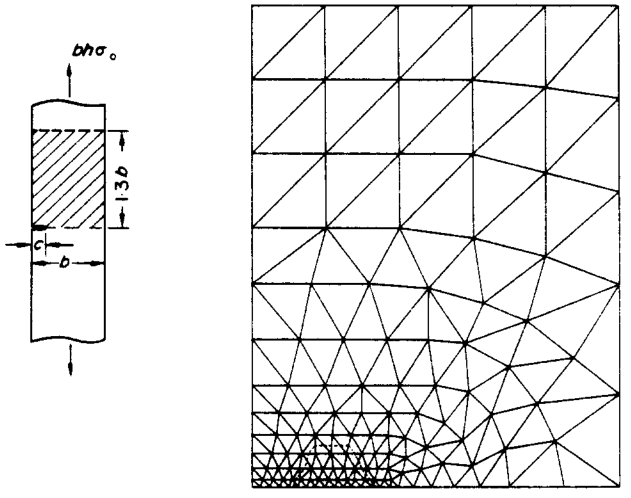


FIG. 3. Uniaxial stretching of a nicked strip and division into finite elements. Dotted contour indicates a typical path for calculating the Rice integral (c/b = 0·2).

In order to evaluate the J integral, it is necessary to calculate the stress and deformation in the strips as functions of the applied load. This is accomplished with the use of a plane-stress finite element procedure for highly elastic materials. The finite element procedure employed in this paper is derived from a variational equation for which the Euler equations are the equilibrium equations for incremental stresses and displacements. Consequently, the stiffness equations are of the incremental form. The details of the procedure are provided in Appendix A. It should be noted that the derivation of the procedure follows closely the approach formulated by Hibbitt *et al.* [15] for problems of finite strain and large displacement. In [15] the virtual velocity theorem is the basis for the transition from continuum theorems to finite element discretization, while in the present derivation the time-derivative version of the virtual velocity theorem is the "crossing over" point. It should also be noted that finite element procedures for the bulging and stretching of elastic sheets have been developed by Oden [9]. Oden's approach is to derive from the minimum theorem of potential energy a system of nonlinear stiffness equations, and then to obtain solutions for these equations by means of an appropriate numerical scheme (e.g. the incremental method or the Newton–Raphson method). Thus, the present procedure as described in Appendix A may be viewed as an alternative form of Oden's incremental procedure.

Using the finite element grid shown in Fig. 3, finite element calculations are made for the stress and deformation in the strip by gradually incrementing the applied load. The load is described by the stress σ_0 which is the total applied load divided by the undeformed cross-sectional area of the strip. For a typical incremental step of calculation, the master stiffness matrix, K, is first formed by assembling the elemental stiffness matrix (A.13). The resulting matrix equations have the following form:

$$K\Delta\mathbf{U} = \Delta\sigma_0\mathbf{F} \tag{3.1}$$

where $\Delta\mathbf{U}$ denotes the vector of all nodal displacement increments, $\Delta\sigma_0$ is a pre-assigned load increment and $\mathbf{F}$ is a fixed load-distribution vector, corresponding to the uniaxial stretching. Equations (3.1) are then solved for $\Delta\mathbf{U}$ by a Cholesky factorization technique [16]. Adding $\Delta\mathbf{U}$ to the previously calculated displacements gives the new cumulative nodal displacement vector $\mathbf{U}$, which can now be used to calculate the stresses and strains in the strip. Based on these results, the Rice integral J is then evaluated according to equation (2.3) by following a contour as shown in Fig. 3. The contour is taken in the same way as in [12] such that it enters into and exits from an element at the mid-points of two prescribed sides. In the interior of the element, the contour is a straight line.

To select the load increment, three sizes of $(\Delta\sigma_0/C_1)$, 0·4, 0·2 and 0·1, were tested for the case of $\alpha = 0{\cdot}9$ and $c/b = 0{\cdot}2$. The computed values of J are shown in Fig. 4. It is found that the J values for $\Delta\sigma_0/C_1 = 0{\cdot}4$ and for $\Delta\sigma_0/C_1 = 0{\cdot}2$ are within 6 per cent of each

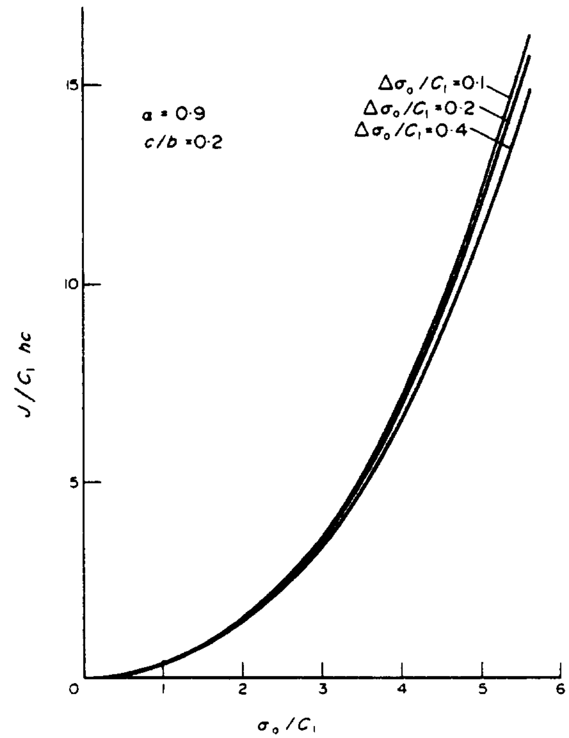


FIG. 4. The calculated Rice integral J vs the stress σ_0 for several sizes of load increment $\Delta\sigma_0$ ($\alpha = 0{\cdot}9$, $c/b = 0{\cdot}2$).

other in the range of $\sigma_0/C_1 < 5$. When the J values for $\Delta\sigma_0/C_1 = 0{\cdot}2$ are compared with those for $\Delta\sigma_0/C_1 = 0{\cdot}1$, the differences are reduced to within 3 per cent. From this numerical experiment, it appears that to compute J to an acceptable accuracy the increment $\Delta\sigma_0/C_1 = 0{\cdot}2$ would be satisfactory and this is used in all the remaining calculations.

The computed values of (J/C_1hc) for vulcanizate A for which the material constant $\alpha = 0{\cdot}9$, are plotted in Fig. 5 against σ_0/C_1 for several ratios of c/b. The results show that, for a relatively small cut, say $c/b < 0{\cdot}2$, the curve corresponding to $c/b = 0{\cdot}1$ may be adequate for estimating J for a given σ_0/C_1. Similar results were obtained with vulcanizates D ($\alpha = 0{\cdot}74$) and E ($\alpha = 0{\cdot}54$). In Fig. 6, we have plotted the J values for the three vulcanizates for $c/b = 0{\cdot}1$.

4. COMPARISON OF CALCULATED RESULTS WITH EXPERIMENT

Using the critical energy characteristic, T_c, of the vulcanizates A, D and E (given in Table 4 of [1]) and identifying the corresponding critical value of the Rice integral J_c by

$$J_c = T_c h,$$

we can then calculate from Fig. 6 the critical load, σ_c, that causes the growth of a cut, as a function of the cut length c. These critical loads are plotted in Figs. 7(a–c) for the vulcanizates A, D and E, respectively. The experimental data reported by Rivlin and Thomas [1]

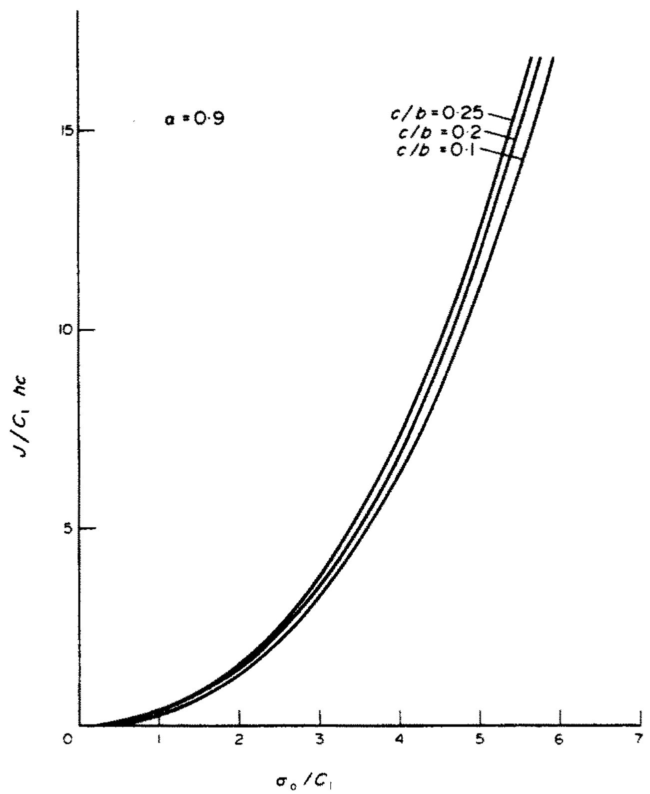


FIG. 5. The calculated Rice integral J vs the stress σ_0 for $\alpha = 0{\cdot}9$ and for $c/b = 0{\cdot}1$, $0{\cdot}2$ and $0{\cdot}25$.

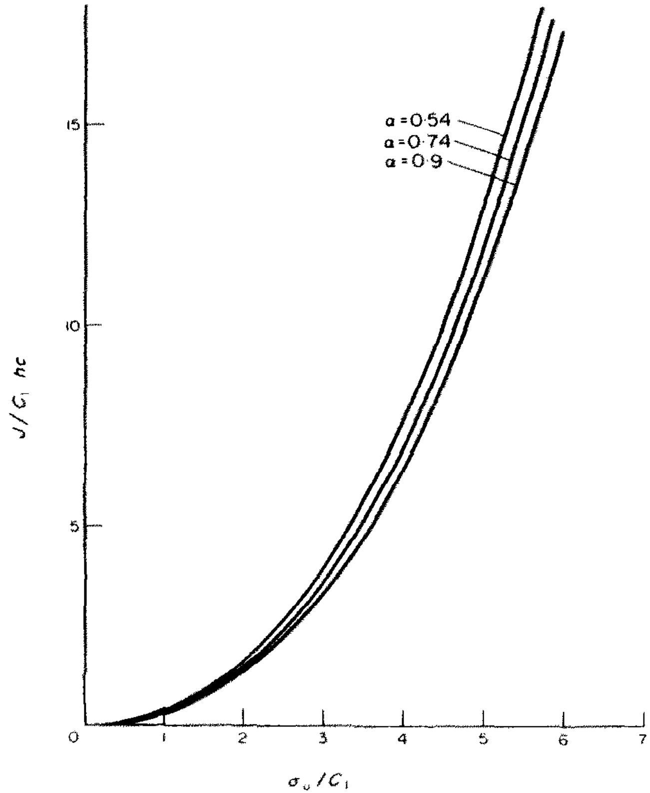


FIG. 6. The calculated Rice integral J vs the stress σ_0 for $c/b = 0{\cdot}1$ and for $\alpha = 0{\cdot}54$, $0{\cdot}74$ and $0{\cdot}9$.

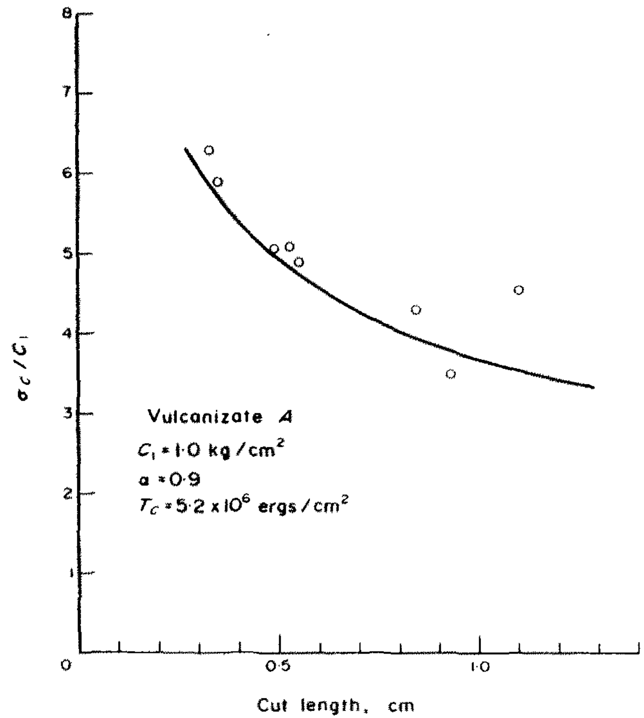
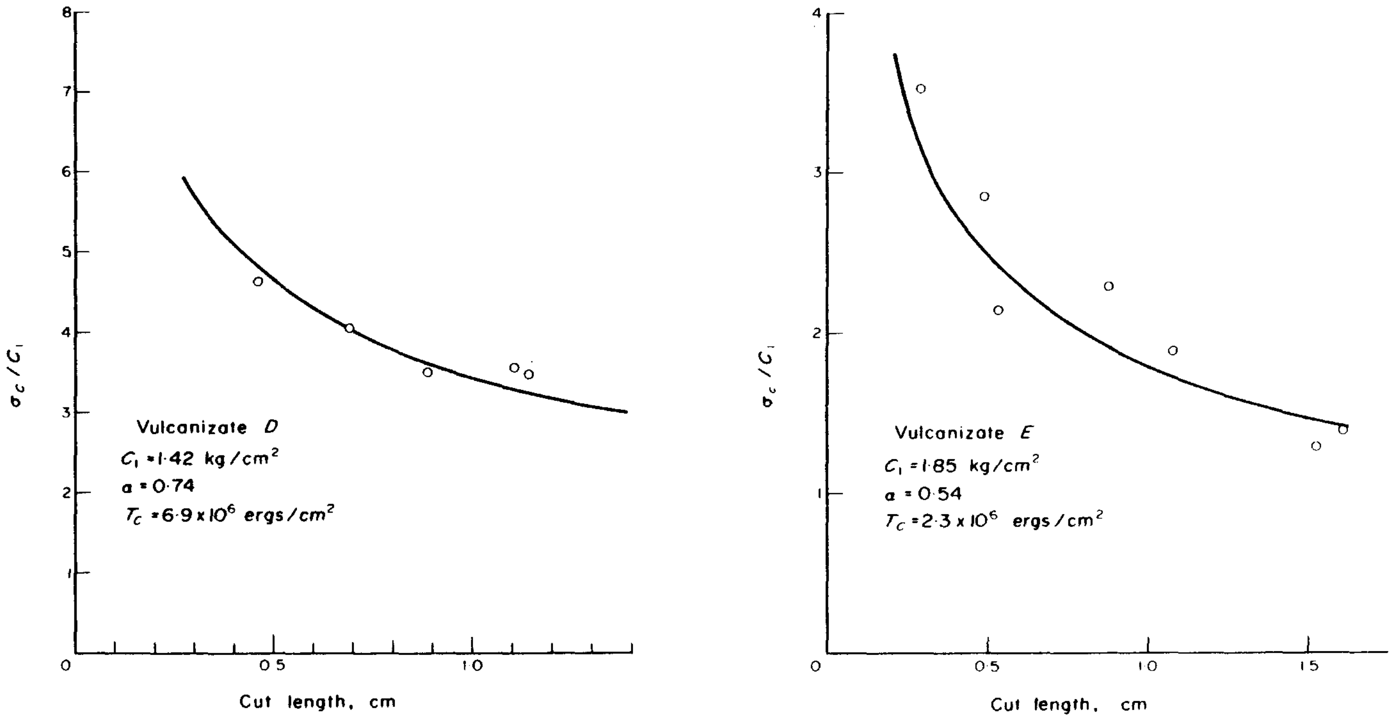


FIG. 7(a).



FIGS. 7(b) and (c). Comparison of the calculated critical stress σ_c (referred to the undeformed test piece) with experimental data for vulcanizate (a) *A*, (b) *D* and (c) *E*.

have also been plotted in these figures, as represented by circles. It is seen that the calculated results agree fairly closely with the experimental data. It should also be noted that the cut lengths of the strips investigated by Rivlin and Thomas are small compared to the width of the strip. In all but three cases, the ratio c/b is less than 0·2.

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APPENDIX A

A plane-stress finite element procedure

Consider a flat sheet whose material coordinates are (x^1, x^2) of a plane cartesian coordinate system. The sheet occupies a domain $\mathscr{D}$ with a boundary $\partial\mathscr{D}$. Under plane loadings, the sheet deforms to a new configuration which is completely described by the displacement u_i $(i = 1, 2)$. The variational equation from which we derive our incremental stiffness is as follows (e.g. see [17]):

$$\int_{\mathscr{D}} \dot{n}^{ij}\delta\dot{\varepsilon}_{ij}\,\mathrm{d}A + \int_{\mathscr{D}} n^{ij}\delta[\tfrac{1}{2}(\dot{u}^k_{,i}\dot{u}_{k,j})]\,\mathrm{d}A = \int_{\partial\mathscr{D}} \dot{F}^i\delta\dot{u}_i\,\mathrm{d}s, \tag{A.1}$$

where a dot over a symbol denotes increment or the rate of change vs "time". (For simplicity, we have assumed that all external loads are prescribed on $\partial\mathscr{D}$ only). In equation (A.1), n^{ij} are Kirchhoff stress resultant tensors on convected coordinates, F^i are the forces per unit undeformed length along $\partial\mathscr{D}$; and

$$\dot{\varepsilon}_{ij} = \dot{e}_{ij}+\tfrac{1}{2}(u^k_{,i}\dot{u}_{k,j}+\dot{u}^k_{,i}u_{k,j}), \tag{A.2}$$

$$\dot{e}_{ij} = \tfrac{1}{2}(\dot{u}_{i,j}+\dot{u}_{j,i}). \tag{A.3}$$

Let $\mathscr{D}$ be divided into triangular elements, and let the approximating functions over each element be linear in x^1 and x^2. We then have, for a typical element,

$$\dot{\mathbf{e}} \equiv \begin{bmatrix} \dot{e}_{11} \\ \dot{e}_{22} \\ 2\dot{e}_{12} \end{bmatrix} = H\dot{\mathbf{u}}, \tag{A.4}$$

and

$$\dot{\boldsymbol{\varepsilon}} \equiv \begin{bmatrix} \dot{\varepsilon}_{11} \\ \dot{\varepsilon}_{22} \\ 2\dot{\varepsilon}_{12} \end{bmatrix} = (H+\varphi B)\dot{\mathbf{u}}, \tag{A.5}$$

where the vector $\dot{\mathbf{u}}$ represents the usual 6-dimensional, nodal displacement increments. The matrices H, B and φ are defined in terms of the initial nodal coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$ and the current spatial derivatives of displacements as follows:

$$B = \frac{1}{\Delta}\begin{bmatrix} (y_2-y_3) & 0 & (y_3-y_1) & 0 & (y_1-y_2) & 0 \\ -(x_2-x_3) & 0 & -(x_3-x_1) & 0 & -(x_1-x_2) & 0 \\ 0 & (y_2-y_3) & 0 & (y_3-y_1) & 0 & (y_1-y_2) \\ 0 & -(x_2-x_3) & 0 & -(x_3-x_1) & 0 & -(x_1-x_2) \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.B,$$

and

$$\varphi = \begin{bmatrix} \dfrac{\partial u}{\partial x} & 0 & \dfrac{\partial v}{\partial x} & 0 \\ 0 & \dfrac{\partial u}{\partial y} & 0 & \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial x} \end{bmatrix},$$

where $\Delta = x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)$. Here, we have temporarily used the notation $(x, y) \equiv (x^1, x^2)$ and $(u, v) \equiv (u_1, u_2)$. The subscripts refer to the nodes of the element. By denoting $\mathbf{n}^T = (n^{11}, n^{22}, n^{12})$ and defining

$$N = \begin{bmatrix} n^{11} & n^{12} & 0 & 0 \\ n^{12} & n^{22} & 0 & 0 \\ 0 & 0 & n^{11} & n^{12} \\ 0 & 0 & n^{12} & n^{22} \end{bmatrix}, \tag{A.6}$$

the elemental equilibrium equations can be derived from equation (A.1) to give

$$A\{(H+\varphi B)^T\dot{\mathbf{n}}+(B^TNB)\dot{\mathbf{u}}\} = \dot{\mathbf{f}}, \tag{A.7}$$

where A is the undeformed elemental area, and $\dot{\mathbf{f}}$ is the vector of nodal force increments corresponding to $\dot{\mathbf{u}}$. Equations (A.7) are valid whether the material is elastic or inelastic, compressible or incompressible.

We now assume that the material of the sheet is incompressible and that there exists a strain energy function W such that the Kirchhoff stresses s^{ij} satisfy

$$s^{ij} = \frac{1}{2}\left(\frac{\partial W}{\partial \varepsilon_{ij}}+\frac{\partial W}{\partial \varepsilon_{ji}}\right), \qquad (i, j = 1, 2). \tag{A.8}$$

(It should be noted that for three-dimensional incompressible solids, Expression (A.8) usually contains an additional term representing the hydrostatic pressure. However, under the present two-dimensional plane stress assumption, each individual component of ε_{ij} $(i, j = 1, 2)$ can vary independently and, hence, equation (A.8) is a valid expression.) By definition, the Kirchhoff stress resultants are simply

$$n^{ij} = hs^{ij}, \tag{A.9}$$

where h is the undeformed thickness of the sheet. The increments of n^{ij} can be obtained by differentiating (A.9) to give

$$\dot{n}^{ij} = h\frac{\partial}{\partial \varepsilon_{kl}}\left[\frac{1}{2}\left(\frac{\partial W}{\partial \varepsilon_{ij}}+\frac{\partial W}{\partial \varepsilon_{ji}}\right)\right]\dot{\varepsilon}_{kl}. \tag{A.10}$$

In matrix notation, equation (A.10) can be written as

$$\dot{\mathbf{n}} = Q\dot{\boldsymbol{\varepsilon}}, \tag{A.11}$$

where Q is a 3×3 matrix. Substituting (A.11) into (A.7) and making use of (A.5) gives the elemental stiffness matrix equations:

$$K_e\dot{\mathbf{u}} = \dot{\mathbf{f}}, \tag{A.12}$$

where

$$K_e = A\{(H+\varphi B)^T Q(H+\varphi B)+B^TNB\}. \tag{A.13}$$

Let the strain energy function of the material be of the Mooney form, i.e.,

$$W = C_1[(I_1-3)+\alpha(I_2-3)], \tag{A.14}$$

where C_1 and α are material constants and I_1, I_2 are strain invariants, defined in terms of the deformed metric tensors, G_{ij}, G^{ij} and $G(=\det(G_{ij}))$ by

$$\begin{aligned} I_1 &= 1/G+G_{11}+G_{22}, \\ I_2 &= G+G^{11}+G^{22}. \end{aligned} \tag{A.15}$$

The covariant metric tensors G_{ij} are calculated according to $G_{ij} = \delta_{ij}+2\varepsilon_{ij}$, where ε_{ij} are given by equation (2.6). The contravariant tensors G^{ij} satisfy $G^{ik}G_{kj} = \delta^i_j$. For Mooney materials, the stress–strain relations (A.8) become

$$\begin{aligned} s^{11} &= 2C_1\left\{(1-\beta G^{11})+\alpha\left[\beta+\left(\frac{1}{\beta}-\beta\mu\right)G^{11}\right]\right\}, \\ s^{22} &= 2C_1\left\{(1-\beta G^{22})+\alpha\left[\beta+\left(\frac{1}{\beta}-\beta\mu\right)G^{22}\right]\right\}, \\ s^{12} &= 2C_1\left\{-\beta G^{12}+\alpha\left(\frac{1}{\beta}-\beta\mu\right)G^{12}\right\}, \end{aligned} \tag{A.16}$$

where $\mu = G^{11}+G^{22}$ and $\beta = 1/G$. It follows from equations (A.9) and (A.10) that the matrix Q in equation (A.11) can now be expressed as

$$Q = C_1[A_1+\alpha A_2],$$

where

$$A_1 = h\beta\begin{bmatrix} 8(G^{11})^2 & 4[G^{11}G^{22}+(G^{12})^2] & 8G^{11}G^{22} \\ & 8(G^{22})^2 & 8G^{12}G^{22} \\ \text{Symmetrical} & & 2[G^{11}G^{22}+3(G^{12})^2] \end{bmatrix}, \tag{A.17}$$

and

$$A_2 = h\begin{bmatrix} 8G^{11}(\mu G^{11}-\beta) & 4[1+2\mu(G^{12})^2] & 4G^{12}(2\mu G^{11}-\beta) \\ & 8G^{22}(\mu G^{22}-\beta) & 4G^{12}(2\mu G^{22}-\beta) \\ \text{Symmetrical} & & 2[\mu\beta+4\mu(G^{12})^2-1] \end{bmatrix}. \tag{A.18}$$

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Абстракт—Обсуждается критерий Ривлина-Томаса для задачи разрывания вулканизатов и резины, в виде интеграла Райса, независимого от траектории. Доказывается что критерий Ривлина-Томаса можно формулировать в альтернативной форме, в которой рост среза определяется с помощью этого интеграла. Затем, используется критерий Райса для предсказания критической нагрузки, которая является причиной роста среза, для случая одноосного растяжения полос с зарубкой, изготовленных из резинных вулканизатов. С помощью метода конечного элемента для чрезвычайно упругих материалов, выполняется расчет интеграла Райса. Указано что предсказанные критические нагрузки согласовываются совершенно близко с экспериментальными данными.